

Equivariant Variational Symbols on Matrix Fibers: Exact Spectra, Rigidity, and Four-Dimensional Normal Forms

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Abstract

Let $\mathfrak{u}(N)$ carry the real Hilbert–Schmidt pairing. We classify the real-form-preserving products $aPY + bYP$ and compute the complete singular spectrum, determinant, and kernel of the resulting commutator–Jordan pencil for an arbitrary skew-Hermitian coefficient. Since the commutator channel is skew-adjoint and the Jordan channel is self-adjoint, the principal coefficient of an undoubled first-order Hessian loses the elliptic Jordan component and retains a central kernel.

We then classify all real bilinear $U(N)$ -equivariant laws on $\mathfrak{u}(N)$. A coefficient–subspace rigidity theorem shows that imposing skew-adjointness only along the actual linear coefficient image yields no additional channels: every induced symbol is a commutator plus one canonical scalar–traceless wedge. Such a symbol is elliptic if and only if $N = 2$, both coefficients are nonzero, and the traceless coefficient map into $\mathfrak{su}(2)$ is injective. In Pauli coordinates the exceptional family is a weighted div–grad–curl system; its Pfaffian sign distinguishes left- and right-quaternionic chirality.

Independently of equivariance and bilinearity, we classify every maximal skew-adjoint elliptic pencil $A : \mathbb{R}^3 \rightarrow \mathfrak{so}(4)$. Hodge decomposition gives an exact repeated singular spectrum and inverse, identifies each Pfaffian sector with graphs of strict contractions, and yields a deformation retraction to chiral Clifford pencils. The definite Pfaffian form supplies a canonical base metric, sharp coercivity and condition-number constants, a signed Cartan normal form, and a Pfaffian-preserving geodesic flattening whose worst condition number obeys an exact exponential law. We place these results within the Hurwitz–Radon bound and determine sharp pointwise centralizer repairs.

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1 Introduction

A linear differential equation obtained as an Euler–Lagrange equation has a formally self-adjoint linearization. This is the Helmholtz necessary condition in the inverse problem of the calculus of variations [8, 4]. For a real first-order operator, formal self-adjointness makes the coefficient matrices skew-adjoint, whereas ellipticity requires every nonzero covector combination to be invertible. Generalized gradients and Clifford operators show that these requirements can coexist in suitable representation-theoretic architectures [5, 10, 7]. The question here is what remains when the fiber is

the real Hilbert space $\mathfrak{u}(N)$ and the coefficients are produced from one matrix-valued linear map by a conjugation-equivariant bilinear law.

The first part of the paper resolves this question. We begin with the universal two-word product $aPY + bYP$ on the skew-Hermitian real form. The real-form condition leaves a commutator channel and i times an anticommutator channel. These have opposite adjoint parity. We compute the complete real singular spectrum for an arbitrary skew-Hermitian coefficient J , including the determinant and all kernel multiplicities. The Helmholtz condition removes the Jordan channel, while the surviving commutator always annihilates the scalar direction. This gives an undoubled first-order variational–ellipticity obstruction without any assumption on which spectral branches occur. Thus the two-word and equivariant hypotheses are not merely technical: they isolate a sharp architectural obstruction that is invisible in the unrestricted Hurwitz–Radon setting.

We next remove the two-word restriction. Classical matrix-concomitant invariant theory reduces every real bilinear $U(N)$ -equivariant law on $\mathfrak{u}(N)$ to six elementary channels [11]. The principal-symbol condition, however, only tests coefficients in the actual image of the covector map. We therefore impose skew-adjointness on an arbitrary linear coefficient subspace rather than on every matrix. An exact block decomposition proves coefficient–subspace rigidity: all extra channels vanish on that subspace or are forced out, leaving only the commutator and one canonical scalar–traceless wedge. Their spectrum gives the complete classification. The induced symbol is elliptic if and only if $N = 2$, both coefficients are nonzero, and the traceless coefficient map into $\mathfrak{su}(2)$ is injective; in particular the base dimension is at most three.

The exceptional rank-two family admits a concrete description. In Pauli coordinates it is a weighted div–grad–curl system. Its Pfaffian sign divides the elliptic coefficient plane into left- and right-quaternionic sectors. Pointwise polar normalization identifies the corresponding chiral factor, while an explicit coefficient homotopy reaches a balanced covector-linear Clifford symbol. The Pauli construction is therefore not an isolated repair: it is the canonical representative of each equivariant elliptic component.

Section 7 then removes equivariance and bilinearity entirely and places this exception inside the unrestricted real four-dimensional problem. The Hurwitz–Radon theorem gives the general base-dimension bound for skew-adjoint elliptic pencils [1, 2, 3]. In the maximal case $A : \mathbb{R}^3 \rightarrow \mathfrak{so}(4)$, Hodge decomposition gives more than a bound. The Pfaffian is the difference of the squared self-dual and anti-self-dual components, so every elliptic pencil is an isomorphism into one Hodge summand plus a strict contraction into the other. From this graph form we derive the repeated singular values, a closed inverse formula, the path components, and a deformation retraction to chiral Clifford pencils. The classical geometry of definite three-planes and noncompact symmetric spaces provides the background [9, 6]; the operator formulas here identify the Pfaffian metric, sharp coercivity, signed Cartan parameters, and a Pfaffian-preserving geodesic flattening with exact condition-number decay.

The two parts meet precisely on $\mathfrak{u}(2)$: the trace-completed Pauli symbol saturates the Hurwitz–Radon bound and represents both universal chiralities. For general J , we also determine the minimum pointwise rank needed to repair the full commutator centralizer. That construction need not vary linearly or smoothly with a covector and is therefore distinct from the equivariant principal-symbol classification. Doubled fields, indefinite or symplectic pairings, constrained systems, higher-order normal equations, and non-equivariant nonlinear coefficient laws lie outside the scope of the paper.

The principal conclusions may be summarized as follows.

- A. Coefficient–subspace rigidity.** Restricting the formal-adjoint condition to the actual linear coefficient image does not create additional equivariant channels: the symbol is forced to be a

commutator plus the canonical scalar–traceless wedge.

- B. Complete matrix-fiber classification.** Such a symbol is elliptic exactly for $N = 2$, with both surviving coefficients nonzero and an injective traceless coefficient map into $\mathfrak{su}(2)$. The balanced members are precisely the two quaternionic Clifford chiralities.
- C. Universal four-dimensional comparison.** Every maximal pencil $\mathbb{R}^3 \rightarrow \mathfrak{so}(4)$ is governed by its definite Pfaffian form and Hodge contraction, admits an exact signed Cartan normal form, and has a canonical Pfaffian-preserving geodesic to its chiral Clifford representative.

The invariant-theory spanning theorem, the Hurwitz–Radon bound, and the geometry of definite three-planes are classical inputs. The new claims are the coefficient-image rigidity result and its exact ellipticity classification, together with the operator-theoretic spectral, metric, and geodesic formulas in the maximal four-dimensional comparison. This separation is maintained throughout the statements and remarks below. Every theorem is proved analytically in the text. The ancillary programs are independent machine checks only and are not used as proof substitutes.

The paper is organized as follows. Sections 2–5 treat two-word products, exact spectra, quantitative separation from the variational locus, and the basic obstruction. Section 6 proves the equivariant and coefficient-subspace classification, computes the surviving trace-completed spectrum before using it, and develops the quaternionic normal form. Section 7 studies unrestricted skew-adjoint elliptic pencils, including the maximal four-dimensional normal forms. Section 8 gives the sharp pointwise centralizer repairs. Section 9 states the precise scope, and Appendix A records standard changes of architecture that evade the obstruction.

2 The universal real-form-preserving product

Let $\mathfrak{u}(N)$ denote the real vector space of skew-Hermitian $N \times N$ matrices, equipped with

$$\langle A, B \rangle = -\operatorname{Re} \operatorname{Tr}(AB).$$

For skew-Hermitian A and B , this is also $\operatorname{Re} \operatorname{Tr}(A^*B)$, the underlying real inner product of the usual complex Hilbert–Schmidt pairing. We write $\|J\|_{\text{HS}} = \langle J, J \rangle^{1/2}$ for this fiber norm and $\|J\|_{\text{op}}$ for the spectral norm of a matrix acting on $\mathbb{C}^N$. Operator norms of endomorphisms of $\mathfrak{u}(N)$ are also taken with respect to the Hilbert–Schmidt inner product; the space on which an operator acts will make the notation unambiguous.

Proposition 1 (classification of universal two-sided products). *Assume $N \geq 2$. A two-word expression*

$$m_{a,b}(P, Y) = aPY + bYP, \quad a, b \in \mathbb{C},$$

maps $\mathfrak{u}(N) \times \mathfrak{u}(N)$ into $\mathfrak{u}(N)$ for all P, Y if and only if $b = -\bar{a}$. Equivalently, there are unique $x, y \in \mathbb{R}$ such that

$$m_{a,b}(P, Y) = m_{x,y}(P, Y) := x[P, Y] + iy(PY + YP).$$

Proof. For skew-Hermitian P, Y ,

$$m_{a,b}(P, Y)^* = \bar{a}YP + \bar{b}PY.$$

Requiring this to equal $-m_{a,b}(P, Y)$ for all P, Y gives $\bar{b} = -a$ and $\bar{a} = -b$. The two word orders can be separated, for example, by skew-Hermitian 2×2 test matrices for which PY and YP are not proportional. Writing $a = x + iy$ gives $b = -x + iy$. $\square$

Fix $J \in \mathfrak{u}(N)$ and define real endomorphisms of $\mathfrak{u}(N)$ by

$$C_J(Y) = [J, Y], \quad S_J(Y) = i(JY + YJ).$$

Lemma 1 (adjoint parity and commutation). *With respect to the real Hilbert–Schmidt pairing,*

$$C_J^* = -C_J, \quad S_J^* = S_J,$$

and

$$C_J S_J = S_J C_J.$$

Proof. Cyclicity of trace gives the two adjoint identities. Writing L_J and R_J for left and right multiplication,

$$C_J = L_J - R_J, \quad S_J = i(L_J + R_J),$$

and $L_J R_J = R_J L_J$, which proves commutation. $\square$

3 Complete spectrum for an arbitrary skew-Hermitian coefficient

Choose a unitary basis in which

$$J = i \operatorname{diag}(\lambda_1, \dots, \lambda_N), \quad \lambda_r \in \mathbb{R}.$$

Unitary conjugation acts orthogonally on $\mathfrak{u}(N)$. Let

$$D_r = iE_{rr},$$

and, for $r < s$,

$$A_{rs} = E_{rs} - E_{sr}, \quad B_{rs} = i(E_{rs} + E_{sr}).$$

After harmless normalization, these vectors form an orthonormal real basis.

Theorem 1 (exact real block decomposition). *Set*

$$M_{x,y}(J) = xC_J + yS_J.$$

Then each $\mathbb{R}D_r$ and each $V_{rs} = \operatorname{span}_{\mathbb{R}}\{A_{rs}, B_{rs}\}$ is invariant. On these blocks,

$$M_{x,y}(J)D_r = -2y\lambda_r D_r,$$

and

$$M_{x,y}(J)|_{V_{rs}} = \begin{pmatrix} -y(\lambda_r + \lambda_s) & -x(\lambda_r - \lambda_s) \\ x(\lambda_r - \lambda_s) & -y(\lambda_r + \lambda_s) \end{pmatrix}.$$

Consequently, the real singular values are

$$2|y\lambda_r|, \quad 1 \leq r \leq N,$$

and

$$\sqrt{x^2(\lambda_r - \lambda_s)^2 + y^2(\lambda_r + \lambda_s)^2}, \quad 1 \leq r < s \leq N,$$

the latter each with multiplicity two. Moreover,

$$\det_{\mathbb{R}} M_{x,y}(J) = (-2y)^N \left(\prod_{r=1}^N \lambda_r \right) \prod_{r < s} [x^2(\lambda_r - \lambda_s)^2 + y^2(\lambda_r + \lambda_s)^2].$$

Proof. Direct multiplication gives

$$C_J A_{rs} = (\lambda_r - \lambda_s) B_{rs}, \quad C_J B_{rs} = -(\lambda_r - \lambda_s) A_{rs},$$

while

$$S_J A_{rs} = -(\lambda_r + \lambda_s) A_{rs}, \quad S_J B_{rs} = -(\lambda_r + \lambda_s) B_{rs}.$$

The displayed 2×2 matrix is a scalar multiple of the identity plus a real rotation generator, so both of its singular values are the stated square root and its determinant is the square of that number. The diagonal blocks give the remaining factors. $\square$

Corollary 1 (invertibility and kernel dimensions). *Let m_λ be the multiplicity of the eigenvalue λ of $-iJ$, and let m_0 be the nullity of J .*

If $x \neq 0$ and $y \neq 0$, then

$$\dim_{\mathbb{R}} \ker M_{x,y}(J) = m_0^2,$$

so $M_{x,y}(J)$ is invertible if and only if J is invertible.

If $y = 0$ and $x \neq 0$, then

$$\dim_{\mathbb{R}} \ker M_{x,0}(J) = \sum_{\lambda} m_{\lambda}^2,$$

the dimension of the unitary centralizer of J .

If $x = 0$ and $y \neq 0$, then

$$\dim_{\mathbb{R}} \ker M_{0,y}(J) = \sum_{\lambda} m_{\lambda} m_{-\lambda}.$$

In particular, the pure Jordan axis is invertible precisely when $\text{spec}(-iJ) \cap (-\text{spec}(-iJ)) = \emptyset$.

Proof. The block theorem shows that a mixed off-diagonal block can vanish only when $\lambda_r - \lambda_s = \lambda_r + \lambda_s = 0$, hence only inside the zero eigenspace. The pure commutator kernel consists of all matrices preserving each eigenspace. The pure Jordan kernel consists of all matrix entries joining λ to $-\lambda$. Counting real dimensions gives the formulas. $\square$

Corollary 2 (scalar Clifford specialization). *Assume*

$$J^2 = -\rho^2 I, \quad \rho > 0,$$

and let p and q be the multiplicities of the eigenvalues $+\rho$ and $-\rho$ of $-iJ$. Then the commuting sector has real dimension $p^2 + q^2$, the anticommuting sector has real dimension $2pq$, and

$$M_{x,y}(J)^* M_{x,y}(J) = 4\rho^2 (y^2 \Pi_+ + x^2 \Pi_-).$$

Thus $2\rho|y|$ occurs with multiplicity $p^2 + q^2$ and $2\rho|x|$ with multiplicity $2pq$. In particular,

$$|\det_{\mathbb{R}} M_{x,y}(J)| = (2\rho)^{N^2} |y|^{p^2+q^2} |x|^{2pq}.$$

If $p, q > 0$, the symbol is invertible exactly when $xy \neq 0$; if one sign is absent, it is invertible exactly when $y \neq 0$.

When $p, q > 0$, the branch formula also gives

$$\|M_{x,y}(J)\| = 2\rho \max\{|x|, |y|\}, \quad s_{\min}(M_{x,y}(J)) = 2\rho \min\{|x|, |y|\}.$$

4 Quantitative separation from variational parity

Let $\mathcal{K}$ be the real vector space of skew-adjoint endomorphisms of $\mathfrak{u}(N)$, equipped with the operator norm.

Proposition 2 (exact distance to the skew-adjoint locus). *For arbitrary $J \in \mathfrak{u}(N)$,*

$$\text{dist}(M_{x,y}(J), \mathcal{K}) = \|yS_J\| = 2|y| \|J\|_{\text{op}}.$$

If $M_{x,y}(J)$ is invertible, then

$$\text{dist}(M_{x,y}(J), \mathcal{K}) \geq s_{\min}(M_{x,y}(J)).$$

Proof. For every $K \in \mathcal{K}$, the symmetric part of $M_{x,y} - K$ is yS_J . Since the symmetric-part map is contractive in operator norm,

$$\|M_{x,y} - K\| \geq \|yS_J\|.$$

Taking $K = xC_J$ attains equality. The exact spectrum above gives $\|S_J\| = 2 \max_r |\lambda_r| = 2\|J\|_{\text{op}}$. If $M_{x,y}$ is invertible, then $y \neq 0$ and J is invertible; choosing a diagonal direction associated to an eigenvalue of minimum modulus shows

$$s_{\min}(M_{x,y}) \leq 2|y| \min_r |\lambda_r| \leq 2|y| \|J\|_{\text{op}}.$$

□

Corollary 3 (exact Clifford coefficient-plane geometry). *Under the scalar Clifford hypotheses with $p, q > 0$, let*

$$\Sigma_J = \{M_{x',y'}(J) : x'y' = 0\}.$$

Then

$$\text{dist}(M_{x,y}(J), \Sigma_J) = 2\rho \min\{|x|, |y|\} = s_{\min}(M_{x,y}(J)).$$

Hence every elliptic mixed Clifford symbol satisfies

$$\text{dist}(M_{x,y}(J), \mathcal{K}) \geq s_{\min}(M_{x,y}(J)).$$

Corollary 4 (minimum size of a variational repair). *If $M_{x,y}(J)$ is invertible and R is any endomorphism such that $M_{x,y}(J) + R$ is skew-adjoint, then*

$$\|R\| \geq 2|y| \|J\|_{\text{op}} \geq s_{\min}(M_{x,y}(J)).$$

Every perturbation with norm strictly below $s_{\min}$ preserves invertibility. The smallest repair, $R = -yS_J$, lands on the singular pure commutator axis.

5 The variational–ellipticity obstruction

Let $E \rightarrow M$ be a real metric vector bundle whose fiber is $\mathfrak{u}(N)$, and let

$$L = \sum_{j=1}^n A_j \nabla_{e_j} + B$$

be a real first-order operator. Its coefficient symbol is

$$A(\xi) = \sum_j \xi_j A_j.$$

Lemma 2 (principal formal-adjoint condition). *If L is formally self-adjoint, then*

$$A_j^* = -A_j \quad \text{for every } j, \quad \text{equivalently} \quad A(\xi)^* = -A(\xi).$$

With the alternative complexified convention $\sigma_L(\xi) = iA(\xi)$, the same statement reads $\sigma_L(\xi)^* = \sigma_L(\xi)$.

Proof. Metric integration by parts gives

$$(A_j \nabla_{e_j})^* = -A_j^* \nabla_{e_j} + \text{terms of order zero.}$$

Equality of the first-order coefficients of L and L^* gives the claim. $\square$

Theorem 2 (undoubled variational–ellipticity obstruction). *Suppose the principal coefficient of a real first-order linearization is*

$$A_{x,y}(\xi) = xC_{J(\xi)} + yS_{J(\xi)},$$

where $J(\xi) \in \mathfrak{u}(N)$ depends linearly on ξ . If the linearization is the Hessian of a real local action, then it cannot be elliptic.

Proof. A Hessian is formally self-adjoint, so its coefficient symbol is skew-adjoint. The adjoint-parity lemma gives

$$A_{x,y}(\xi)^* = -xC_{J(\xi)} + yS_{J(\xi)}.$$

Thus $yS_{J(\xi)} = 0$ for every ξ . If $J(\xi_0) = 0$ for some nonzero ξ_0 , then $A_{x,y}(\xi_0) = 0$ and ellipticity already fails. Otherwise $S_{J(\xi)}$ is nonzero because

$$S_{J(\xi)}(iI) = -2J(\xi),$$

so $y = 0$. The remaining commutator symbol satisfies

$$C_{J(\xi)}(iI) = 0$$

for every ξ , and is therefore not invertible. $\square$

Corollary 5 (Clifford coefficient symbols). *If $J(\xi)^2 = -|\xi|^2 I$, then every genuinely mixed coefficient with $x \neq 0$ and $y \neq 0$ is elliptic. If both Clifford signs occur, its ellipticity margin is*

$$2|\xi| \min\{|x|, |y|\}.$$

Nevertheless it cannot be the principal coefficient of an undoubled real first-order Hessian. The variationally admissible pure commutator axis has a central kernel.

Corollary 6 (lower-order corrections). *No zeroth-order scalar, finite-rank, or other lower-order correction can restore principal ellipticity to the commutator axis or principal formal self-adjointness to a mixed axis.*

6 Complete equivariant and coefficient–subspace classification

The two-word theorem leaves open whether another natural bilinear matrix law could restore variational ellipticity. We now classify all such laws under the basic symmetry of simultaneous unitary conjugation.

Write

$$e = \frac{iI}{\sqrt{N}}, \quad \tau(P) = \langle e, P \rangle, \quad P_0 = P - \tau(P)e,$$

and define the traceless Jordan channel

$$D_P(Y) = i(P_0 Y_0 + Y_0 P_0) + \frac{2}{\sqrt{N}} \langle P_0, Y_0 \rangle e.$$

The correction makes $D_P(Y)$ traceless. For $N = 2$ this channel vanishes identically by the polarized Cayley–Hamilton identity.

Theorem 3 (bilinear conjugation-equivariant basis). *Let $N \geq 2$, and let*

$$B : \mathfrak{u}(N) \times \mathfrak{u}(N) \longrightarrow \mathfrak{u}(N)$$

be real bilinear and equivariant under simultaneous unitary conjugation. If $N \geq 3$, there are unique real coefficients a, b, c, d, f, g such that

$$\begin{aligned} B(P, Y) = & a[P, Y] + bD_P(Y) + c\tau(P)Y_0 + d\tau(Y)P_0 \\ & + f\langle P_0, Y_0 \rangle e + g\tau(P)\tau(Y)e. \end{aligned}$$

For $N = 2$ the same statement holds after deleting the identically zero D channel; the remaining five channels are independent.

Proof. Complexify $\mathfrak{u}(N)$ to $M_N(\mathbb{C})$. Since $U(N)$ is Zariski dense in $GL_N(\mathbb{C})$, the complexification of B is a degree-(1, 1) matrix concomitant. The scalar–traceless decomposition

$$M_N(\mathbb{C}) = \mathbb{C}I \oplus \mathfrak{sl}_N$$

also makes the channel count transparent. There is one equivariant map in each of

$$\mathbb{C} \otimes \mathbb{C} \longrightarrow \mathbb{C}, \quad \mathbb{C} \otimes \mathfrak{sl}_N \longrightarrow \mathfrak{sl}_N, \quad \mathfrak{sl}_N \otimes \mathbb{C} \longrightarrow \mathfrak{sl}_N,$$

one invariant pairing $\mathfrak{sl}_N \otimes \mathfrak{sl}_N \rightarrow \mathbb{C}$, and, for $N \geq 3$, two adjoint-valued maps $\mathfrak{sl}_N \otimes \mathfrak{sl}_N \rightarrow \mathfrak{sl}_N$, represented by the bracket and the traceless Jordan product. For $N = 2$ the symmetric adjoint-valued channel vanishes by polarized Cayley–Hamilton, leaving five channels. Equivalently, the first fundamental theorem for matrix concomitants gives the six spanning expressions

$$PY, \quad YP, \quad \text{Tr}(P)Y, \quad \text{Tr}(Y)P, \quad \text{Tr}(PY)I, \quad \text{Tr}(P)\text{Tr}(Y)I$$

[11]. Equivalently, in bidegree (1, 1) the invariant-tensor proof reduces to index contractions: the two open word orderings, the two one-trace routings, the closed pairing trace, and the double trace are exactly the six possible contraction channels. Requiring the skew-Hermitian real form and rewriting with e , P_0 , and Y_0 gives the displayed real basis.

For completeness, independence in degree (1, 1) is elementary. Suppose a linear combination of the six complex concomitants vanishes. For $N \geq 3$, the choices $(P, Y) = (E_{12}, E_{23})$ and (E_{23}, E_{12}) isolate

the coefficients of PY and YP . The choices (E_{11}, E_{23}) and (E_{23}, E_{11}) then isolate the coefficients of $\text{Tr}(P)Y$ and $\text{Tr}(Y)P$. Finally, (E_{12}, E_{21}) isolates $\text{Tr}(PY)I$, and (I, I) isolates $\text{Tr}(P)\text{Tr}(Y)I$.

For $N = 2$ the polarized Cayley–Hamilton relation

$$PY + YP - \text{Tr}(P)Y - \text{Tr}(Y)P + (\text{Tr}(P)\text{Tr}(Y) - \text{Tr}(PY))I = 0$$

removes the traceless Jordan channel. After eliminating this relation, the remaining five channels are independent: (E_{12}, E_{21}) separates the commutator from the scalar trace-pairing channel, (E_{11}, E_{12}) and (E_{12}, E_{11}) isolate the two one-trace channels, and (I, I) isolates the scalar–scalar channel. $\square$

For fixed P , set

$$U_P(Y) = \tau(P)Y_0, \quad V_P(Y) = \tau(Y)P_0, \quad W_P(Y) = \langle P_0, Y_0 \rangle e, \quad Z_P(Y) = \tau(P)\tau(Y)e.$$

Lemma 3 (adjoint table). *For every $P \in \mathfrak{u}(N)$,*

$$\begin{aligned} C_P^* &= -C_P, & D_P^* &= D_P, & U_P^* &= U_P, \\ V_P^* &= W_P, & W_P^* &= V_P, & Z_P^* &= Z_P. \end{aligned}$$

Consequently, a bilinear equivariant law has skew-adjoint coefficient operators $B_P : Y \mapsto B(P, Y)$ for every P if and only if

$$\boxed{B(P, Y) = x[P, Y] + \alpha K_{P_0}(Y)}$$

for unique $x, \alpha \in \mathbb{R}$, where

$$K_{P_0}(Y) = \langle P_0, Y \rangle e - \tau(Y)P_0.$$

Proof. The adjoint identities follow directly from cyclicity of trace and from the orthogonal decomposition $\mathfrak{u}(N) = \mathbb{R}e \oplus \mathfrak{su}(N)$. In the basis theorem, every self-adjoint channel must have zero coefficient, while the coefficients of V_P and W_P must be opposite. This leaves precisely the commutator and scalar–traceless wedge. $\square$

Lemma 4 (rigidity of the traceless Jordan channel). *Assume $N \geq 3$. If $X \in \mathfrak{su}(N)$ and*

$$D_X = \lambda I_{\mathfrak{su}(N)}$$

for some real λ , then $X = 0$ and $\lambda = 0$.

Proof. Diagonalize $X = i \text{diag}(\lambda_1, \dots, \lambda_N)$ with $\sum_r \lambda_r = 0$. On the real off-diagonal two-plane belonging to a pair $r < s$, the operator D_X is scalar multiplication by $-(\lambda_r + \lambda_s)$. If D_X is a scalar identity, all pairwise sums $\lambda_r + \lambda_s$ are equal. For $N \geq 3$ this forces all λ_r to be equal, and the trace-zero condition then gives $\lambda_r = 0$ for every r . Hence $X = 0$ and the scalar is zero. The argument does not require a simple spectrum; it uses only the pairwise-sum condition available when $N \geq 3$. $\square$

Theorem 4 (coefficient–subspace rigidity). *Let $S \subset \mathfrak{u}(N)$ be a real linear subspace, and let B be a real bilinear $U(N)$ -equivariant law. Suppose that*

$$B_P : Y \mapsto B(P, Y)$$

is skew-adjoint for every $P \in S$. Write $S_0 = \{P_0 : P \in S\} \subset \mathfrak{su}(N)$.

If $S_0 = \{0\}$, then $B_P = 0$ for every $P \in S$. If $S_0 \neq \{0\}$, there are unique real numbers x, α such that, for every $P \in S$,

$$\boxed{B(P, Y) = x[P_0, Y] + \alpha K_{P_0}(Y)}.$$

Thus requiring adjoint parity only on a selected coefficient subspace does not enlarge the induced skew-adjoint family.

Proof. Use the six-channel expansion, with the bD_P term understood to be absent when $N = 2$,

$$B_P = aC_P + bD_P + cU_P + dV_P + fW_P + gZ_P.$$

With respect to the orthogonal decomposition $\mathfrak{u}(N) = \mathbb{R}e \oplus \mathfrak{su}(N)$, write $P = te + X$. The symmetric part has the block form

$$\frac{1}{2}(B_P + B_P^*) = \begin{pmatrix} gt & \frac{d+f}{2}X^\flat \\ \frac{d+f}{2}X & bD_X + ctI_{\mathfrak{su}(N)} \end{pmatrix}.$$

Geometrically, the two off-diagonal blocks are the same rank-one coupling between the scalar line and the direction X ; for $X \neq 0$ their vanishing forces $d + f = 0$. The lower-right block must then vanish on the entire traceless summand, and the preceding lemma prevents a nonzero Jordan channel from being hidden inside a scalar multiple of the identity.

If $S_0 = \{0\}$, every $P \in S$ is scalar. The vanishing of the two diagonal blocks removes the only channels that can survive on scalar coefficients, so $B_P = 0$ on S .

Now assume $S_0 \neq \{0\}$ and choose $P = te + X \in S$ with $X \neq 0$. The off-diagonal block gives $d + f = 0$. For $N = 2$, D_X vanishes identically, so the lower-right block gives $ct = 0$. For $N \geq 3$, if $b \neq 0$, the same block would make D_X a scalar identity, contradicting the preceding lemma. Therefore $b = 0$, and again $ct = 0$. The scalar block gives $gt = 0$. Applying the same block equations to every other element of S shows that the cU and gZ channels either have zero coefficients or vanish identically on S . Since $f = -d$, the remaining cross term is $f(W_P - V_P) = fK_{P_0}$. The commutator also depends only on P_0 , proving the displayed formula. Uniqueness follows because K_X couples the scalar line to X , whereas C_X preserves the traceless summand and is nonzero for nonzero traceless X . $\square$

6.1 The surviving trace-completed family

For $J \in \mathfrak{u}(N)$ write

$$J_0 = J - \tau(J)e$$

and define the canonical scalar–traceless wedge

$$K_{J_0}(Y) = \langle J_0, Y \rangle e - \tau(Y)J_0.$$

On skew-Hermitian matrices, $\langle J_0, Y \rangle = -\text{Tr}(J_0 Y)$.

Lemma 5 (canonical scalar–traceless coupling). *The operator K_{J_0} is skew-adjoint, has range contained in $\text{span}_{\mathbb{R}}\{e, J_0\}$, and vanishes on the orthogonal complement of that plane. If $J_0 \neq 0$, then*

$$\text{rank } K_{J_0} = 2$$

and its two nonzero singular values are both $\|J_0\|_{\text{HS}}$. Moreover,

$$C_J K_{J_0} = K_{J_0} C_J = 0.$$

Proof. The formula is the exterior product $e \otimes J_0^\flat - J_0 \otimes e^\flat$, hence is skew-adjoint. It sends e to $-J_0$ and $J_0/\|J_0\|_{\text{HS}}$ to $\|J_0\|_{\text{HS}}e$, and it vanishes on the orthogonal complement. Since e and J_0 commute with J , both belong to $\ker C_J$. Skew-adjointness of C_J then implies that C_J preserves their orthogonal complement, proving both zero-product identities. $\square$

Consider the minimally trace-completed skew-adjoint family

$$A_{x,\alpha}(J) = xC_J + \alpha K_{J_0}, \quad x, \alpha \in \mathbb{R}.$$

Theorem 5 (exact spectrum of the minimal trace completion). *Assume $J_0 \neq 0$, and let $i\lambda_1, \dots, i\lambda_N$ be the eigenvalues of J . The real singular spectrum of $A_{x,\alpha}(J)$ consists of*

$$|\alpha| \|J_0\|_{\text{HS}} \quad \text{with multiplicity two,}$$

$$|x| |\lambda_r - \lambda_s| \quad \text{with multiplicity two for every } r < s,$$

and 0 with an additional multiplicity $N - 2$. These additional zero modes are the diagonal directions orthogonal to $\text{span}\{e, J_0\}$. In particular, if $x\alpha \neq 0$, then

$$\dim_{\mathbb{R}} \ker A_{x,\alpha}(J) = \sum_{\lambda} m_{\lambda}^2 - 2.$$

Hence $A_{x,\alpha}(J)$ is invertible if and only if

$$N = 2, \quad J \text{ is noncentral}, \quad x\alpha \neq 0.$$

Proof. The orthogonal decomposition

$$\mathfrak{u}(N) = \text{span}\{e, J_0\} \oplus \text{span}\{e, J_0\}^{\perp}$$

reduces both summands. On the first plane, $C_J = 0$ and αK_{J_0} is a rotation of modulus $|\alpha| \|J_0\|_{\text{HS}}$. On the orthogonal complement, $K_{J_0} = 0$ and the operator is xC_J . The latter has singular value $|x| |\lambda_r - \lambda_s|$ on each off-diagonal real plane and vanishes on the remaining $N - 2$ diagonal directions. The stated kernel dimension follows from $\dim \ker C_J = \sum_{\lambda} m_{\lambda}^2$. Since $\sum_{\lambda} m_{\lambda}^2 \geq N$, equality with two occurs exactly for a simple two-point spectrum in dimension two. $\square$

Theorem 6 (complete equivariant classification). *Let W be a real n -dimensional vector space, let $J : W \rightarrow \mathfrak{u}(N)$ be linear, and let*

$$A(\xi)Y = B(J(\xi), Y),$$

where B is real bilinear and $U(N)$ -equivariant. Assume only that $A(\xi)$ is skew-adjoint for every $\xi \in W$. Then either $A = 0$, or there are unique $x, \alpha \in \mathbb{R}$ such that

$$A(\xi)Y = x[J_0(\xi), Y] + \alpha K_{J_0(\xi)}(Y).$$

In the nonzero case, $A(\xi)$ is invertible for every $\xi \neq 0$ if and only if

$$\boxed{N = 2, \quad x\alpha \neq 0, \quad J_0 : W \longrightarrow \mathfrak{su}(2) \text{ is injective.}}$$

In particular, every such elliptic symbol has $n \leq 3$.

For $N = 2$ and $J_0(\xi) \neq 0$, its singular values are

$$|\alpha| \|J_0(\xi)\|_{\text{HS}} \quad \text{twice,} \quad \sqrt{2}|x| \|J_0(\xi)\|_{\text{HS}} \quad \text{twice.}$$

Hence, for every $\xi \neq 0$, the pointwise matrix condition number is independent of ξ and of the particular injective map J_0 :

$$\text{cond } A(\xi) = \frac{\max\{|\alpha|, \sqrt{2}|x|\}}{\min\{|\alpha|, \sqrt{2}|x|\}}.$$

A global symbol constant formed by taking a supremum and infimum over a separately normalized unit sphere in W also records the metric distortion of J_0 ; no independence from that additional choice is asserted. The unique balanced ratio, up to signs and common scale, is

$$|\alpha| = \sqrt{2}|x|.$$

At this ratio,

$$A(\xi)^2 = -2x^2 \|J_0(\xi)\|_{\text{HS}}^2 I,$$

and polarization gives

$$A(\xi)A(\eta) + A(\eta)A(\xi) = -2g_J(\xi, \eta)I, \quad g_J(\xi, \eta) = 2x^2 \langle J_0(\xi), J_0(\eta) \rangle.$$

Thus every balanced elliptic member is a real Clifford symbol for the positive-definite metric pulled back by the injective map J_0 .

Proof. Apply the coefficient–subspace rigidity theorem to $S = J(W)$. If $J_0(W) = 0$, then A vanishes. Otherwise the induced family is $A(\xi) = xC_{J_0(\xi)} + \alpha K_{J_0(\xi)}$. The exact trace-completion theorem above shows that this operator is invertible precisely when $N = 2$, $J_0(\xi) \neq 0$, and $x\alpha \neq 0$. Requiring this at every nonzero covector is exactly injectivity of J_0 , which also gives $n \leq \dim \mathfrak{su}(2) = 3$.

For $N = 2$, write the eigenvalues of $-iJ_0$ as $\pm r$. Then $\|J_0\|_{\text{HS}} = \sqrt{2}r$. The commutator plane contributes the repeated singular value $2|x|r = \sqrt{2}|x|\|J_0\|_{\text{HS}}$, while the scalar–traceless plane contributes $|\alpha|\|J_0\|_{\text{HS}}$. This proves the spectrum and the pointwise condition number. At the balanced ratio, the squares on the two reducing planes coincide, giving the square identity. Polarization yields the Clifford relation. $\square$

Remark 1 (scope of the classification). *This theorem does not classify all first-order Hessians. It is sharp within the class of real bilinear $U(N)$ -equivariant coefficient laws whose coefficient map is linear in the covector. Non-equivariant, non-bilinear, doubled, constrained, and higher-order architectures are addressed only as scope boundaries below.*

Proposition 3 (quaternionic and div–grad–curl normal form). *Let*

$$T_a = \frac{i\sigma_a}{\sqrt{2}}, \quad a = 1, 2, 3,$$

where the σ_a are the Pauli matrices, and let

$$\Phi : \mathbb{R} \oplus \mathbb{R}^3 \longrightarrow \mathfrak{u}(2), \quad \Phi(s, v) = se + \sum_{a=1}^3 v_a T_a.$$

This is a real isometry. Write $X = \sum_a j_a T_a$ and $j = (j_1, j_2, j_3)$. Then

$$\boxed{\Phi^{-1}(x[X, \Phi(s, v)] + \alpha K_X(\Phi(s, v))) = (\alpha j \cdot v, -\alpha s j - \sqrt{2}x j \times v).}$$

Consequently, for the full three-dimensional Pauli coefficient map $J_0(\xi) = \sum_a \xi_a T_a$, the corresponding constant-coefficient operator has the exact scalar–vector form

$$\Phi^{-1}L\Phi(s, v) = (\alpha \operatorname{div} v, -\alpha \nabla s - \sqrt{2}x \operatorname{curl} v).$$

Identify $\mathbb{R} \oplus \mathbb{R}^3$ with the quaternions. If $\alpha = \sqrt{2}x$, then the coefficient associated to the imaginary quaternion j is $-\alpha \mathcal{L}_j$, where $\mathcal{L}_j$ denotes left quaternionic multiplication. If $\alpha = -\sqrt{2}x$, it is $-\alpha \mathcal{R}_j$, where $\mathcal{R}_j$ denotes right quaternionic multiplication. In either balanced case, with $\Delta = \sum_{a=1}^3 \partial_a^2$,

$$L^2 = -\alpha^2 \Delta I.$$

Thus the two balanced representatives of the classified equivariant elliptic architecture are not merely abstractly Clifford: after an explicit real isometry they are the left- and right-quaternionic Dirac systems.

Proof. The normalized Pauli basis satisfies

$$[T_a, T_b] = -\sqrt{2} \varepsilon_{abc} T_c.$$

Moreover,

$$K_X(\Phi(s, v)) = (j \cdot v)e - sX.$$

Substitution gives the displayed coordinate formula. Replacing j_a by ∂_a gives the div–grad–curl form. For a pure imaginary quaternion j , left and right multiplication are

$$\mathcal{L}_j(s, v) = (-j \cdot v, sj + j \times v), \quad \mathcal{R}_j(s, v) = (-j \cdot v, sj - j \times v).$$

The two balanced identifications follow immediately. Finally, either from quaternionic multiplication or from $\text{div curl} = 0$, $\text{curl } \nabla = 0$, and $\text{curl curl} = \nabla \text{div} - \Delta$, one obtains $L^2 = -\alpha^2 \Delta I$. $\square$

Remark 2. *This normal form is the point of contact with classical Dirac and generalized-gradient constructions [7, 5]. The quaternionic operator itself is familiar. The new assertion is its uniqueness inside the stated class of real bilinear, conjugation-equivariant first-order Hessian symbols.*

Theorem 7 (Pfaffian chirality and exact spectral flattening). *Retain the scalar–vector coordinates of the preceding proposition and put*

$$\beta = \sqrt{2}x, \quad A_j(s, v) = (\alpha j \cdot v, -\alpha sj - \beta j \times v).$$

Assume $j \neq 0$ and $\alpha\beta \neq 0$. Define the orthogonal reducing planes

$$\mathcal{E}_j = \mathbb{R} \oplus \mathbb{R}j, \quad \mathcal{T}_j = \{0\} \oplus j^\perp,$$

and let $\Pi_{\mathcal{E}_j}$ and $\Pi_{\mathcal{T}_j}$ be their orthogonal projections. Then

$$-A_j^2 = |j|^2 (\alpha^2 \Pi_{\mathcal{E}_j} + \beta^2 \Pi_{\mathcal{T}_j}),$$

so the positive factor in the polar decomposition is

$$|A_j| = (-A_j^2)^{1/2} = |j|(|\alpha| \Pi_{\mathcal{E}_j} + |\beta| \Pi_{\mathcal{T}_j}).$$

With the orientation induced by $(1, e_1, e_2, e_3)$,

$$\boxed{\text{Pf}(A_j) = \alpha\beta|j|^2}, \quad \det A_j = \alpha^2 \beta^2 |j|^4.$$

The orthogonal polar factor $Q_j = A_j |A_j|^{-1}$ is therefore

$$\boxed{Q_j = \begin{cases} -\text{sgn}(\alpha) \mathcal{L}_{\hat{j}}, & \alpha\beta > 0, \\ -\text{sgn}(\alpha) \mathcal{R}_{\hat{j}}, & \alpha\beta < 0, \end{cases}} \quad \hat{j} = j/|j|.$$

Thus spectral flattening forgets the condition-number anisotropy and retains exactly one discrete datum: left versus right quaternionic chirality.

More concretely, set $c = \sqrt{|\alpha\beta|}$ and, for $0 \leq t \leq 1$, define

$$\alpha_t = \operatorname{sgn}(\alpha)((1-t)|\alpha| + tc), \quad \beta_t = \operatorname{sgn}(\beta)((1-t)|\beta| + tc).$$

Then $A_j^{(t)} = A_j(\alpha_t, \beta_t)$ remains invertible for every $j \neq 0$ and every t , while $|\alpha_1| = |\beta_1| = c$. Hence, for any fixed injective coefficient map J_0 , every open (α, β) coefficient quadrant deformation retracts through elliptic symbols to its balanced quaternionic ray. Simultaneous sign reversal sends A to $-A$; after quotienting this two-parameter coefficient family by the global sign, exactly two deformation classes remain, distinguished by $\operatorname{sgn} \operatorname{Pf}(A_j) = \operatorname{sgn}(\alpha x)$. No path of invertible real skew-adjoint coefficients can exchange the two classes, because the Pfaffian cannot change sign without vanishing.

Proof. The planes $\mathcal{E}_j$ and $\mathcal{T}_j$ reduce A_j . On $\mathcal{E}_j$ the operator is a rotation of modulus $|\alpha||j|$, and on $\mathcal{T}_j$ it is a rotation of modulus $|\beta||j|$. This gives the square and positive-factor formulas. In the oriented basis $(1, e_1, e_2, e_3)$ the skew matrix has upper-triangular entries

$$a_{12} = \alpha j_1, \quad a_{13} = \alpha j_2, \quad a_{14} = \alpha j_3, \quad a_{23} = \beta j_3, \quad a_{24} = -\beta j_2, \quad a_{34} = \beta j_1.$$

Therefore

$$\operatorname{Pf}(A_j) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = \alpha\beta|j|^2.$$

Dividing the two reducing rotations by their positive moduli gives the polar factor. The explicit left/right formulas in the preceding proposition then produce the displayed alternatives. Along the stated homotopy neither coefficient vanishes, so the determinant formula keeps the symbol invertible. The final separation follows from continuity of the Pfaffian. $\square$

Corollary 7 (anisotropic square of the div–grad–curl system). *For the full three-dimensional Pauli coefficient map, write $\beta = \sqrt{2}x$. The weighted div–grad–curl operator satisfies*

$$L^2(s, v) = (-\alpha^2 \Delta s, (\beta^2 - \alpha^2) \nabla \operatorname{div} v - \beta^2 \Delta v).$$

Thus scalar and longitudinal modes carry the weight $|\alpha|$, transverse modes carry the weight $|\beta|$, and L^2 is a scalar Laplacian on the entire fiber if and only if $|\alpha| = |\beta|$. The two possible signs at that unique balanced magnitude are precisely the two Pfaffian chiralities.

Proof. Apply L twice and use $\operatorname{div} \operatorname{curl} = 0$, $\operatorname{curl} \nabla = 0$, and $\operatorname{curl} \operatorname{curl} = \nabla \operatorname{div} - \Delta$. $\square$

Corollary 8 (sharp scope of the two-word no-go). *Inside all bilinear conjugation-equivariant coefficient laws, the canonical trace wedge is the only possible skew-adjoint correction to a commutator, even when skew-adjointness is imposed only along the selected coefficient image. It produces an elliptic first-order Hessian symbol only on $\mathfrak{u}(2)$, and every such equivariant elliptic symbol is obtained in this way.*

7 Skew-adjoint elliptic pencils and the maximal four-dimensional case

This section has three layers. First, Hodge splitting represents a maximal pencil as the graph of a strict contraction and yields its Pfaffian, spectrum, inverse, and topology. Second, the definite Pfaffian form supplies the canonical base metric and the signed Cartan invariants. Third, those coordinates

produce the determinant-preserving geodesic to the Clifford locus. The geometry of definite three-planes is classical; the exact operator formulas and their variational-symbol interpretation are the contribution used here.

The preceding obstruction is family-specific. We now record the exact classical restriction on an arbitrary covector-linear skew-adjoint elliptic symbol. If

$$m = (2b + 1)2^{4c+d}, \quad 0 \leq d \leq 3,$$

define the Hurwitz–Radon number by

$$\rho(m) = 8c + 2^d.$$

Theorem 8 (Hurwitz–Radon dimension bound). *Let V be a real inner-product space of dimension m , and let*

$$A : \mathbb{R}^n \longrightarrow \mathfrak{so}(V)$$

be linear. Suppose $A(\xi)$ is invertible for every $\xi \neq 0$. Then

$$n \leq \rho(m) - 1.$$

Conversely, for every $n \leq \rho(m) - 1$ there is such a pencil satisfying the stronger identity

$$A(\xi)^* A(\xi) = |\xi|^2 I_V.$$

Hence $\rho(m) - 1$ is the exact largest base dimension of a real linear skew-adjoint elliptic pencil on an m -dimensional fiber.

Proof. Write $A(\xi) = \sum_{j=1}^n \xi_j A_j$. For $v \in S(V)$ define

$$X_j(v) = A_j v.$$

Skew-adjointness gives $\langle X_j(v), v \rangle = 0$, so each X_j is tangent to the sphere S^{m-1} . If $\sum_j c_j X_j(v) = 0$, then $A(c)v = 0$. The assumed invertibility of $A(c)$ for $c \neq 0$ forces $c = 0$. Thus $X_1, \dots, X_n$ are everywhere linearly independent tangent vector fields. Adams’s theorem gives $n \leq \rho(m) - 1$ [1].

Conversely, the Hurwitz–Radon construction supplies matrices $E_1, \dots, E_{\rho(m)-1} \in \mathfrak{so}(m)$ satisfying

$$E_j E_k + E_k E_j = -2\delta_{jk} I.$$

For any $n \leq \rho(m) - 1$, set $A(\xi) = \sum_{j=1}^n \xi_j E_j$. Then

$$A(\xi)^2 = -|\xi|^2 I, \quad A(\xi)^* A(\xi) = |\xi|^2 I,$$

which proves the converse [2, 3]. □

Theorem 9 (maximal four-dimensional pencil normal form). *Let W and V be oriented Euclidean spaces of dimensions 3 and 4, respectively, and let*

$$A : W \longrightarrow \mathfrak{so}(V)$$

be linear. Use the metric and orientation of V to identify $\mathfrak{so}(V)$ with $\Lambda^2 V^*$ and to split

$$\mathfrak{so}(V) = \mathfrak{so}(V)_+ \oplus \mathfrak{so}(V)_-.$$

Write $A = A_+ + A_-$ for the two Hodge components and normalize their inner product by

$$\langle X, Y \rangle_0 = -\frac{1}{4} \operatorname{tr}(XY).$$

For $\xi \in W$, set

$$r_{\pm}(\xi) = \|A_{\pm}(\xi)\|_0, \quad q_A(\xi) = \operatorname{Pf}(A(\xi)).$$

Then the following statements hold.

1. The Pfaffian and the complete singular spectrum are

$$q_A(\xi) = r_+(\xi)^2 - r_-(\xi)^2,$$

$$s_{\max}(A(\xi)) = r_+(\xi) + r_-(\xi), \quad s_{\min}(A(\xi)) = |r_+(\xi) - r_-(\xi)|,$$

where each singular value occurs with multiplicity two. Whenever $q_A(\xi) \neq 0$, the inverse is

$$A(\xi)^{-1} = -\frac{A_+(\xi) - A_-(\xi)}{q_A(\xi)}, \quad s_{\min}(A(\xi)) = \frac{|q_A(\xi)|}{r_+(\xi) + r_-(\xi)}.$$

Consequently, $A(\xi)$ is invertible for every $\xi \neq 0$ if and only if the quadratic form q_A is positive- or negative-definite.

2. Suppose $q_A > 0$. Then $A_+ : W \rightarrow \mathfrak{so}(V)_+$ is an isomorphism and

$$T = A_- A_+^{-1} : \mathfrak{so}(V)_+ \longrightarrow \mathfrak{so}(V)_-$$

has operator norm strictly smaller than one. Conversely, every pair

$$B \in \operatorname{Iso}(W, \mathfrak{so}(V)_+), \quad \|T\|_{\operatorname{op}} < 1,$$

defines a positive elliptic pencil by

$$A(\xi) = B\xi + TB\xi.$$

Thus the positive sector is smoothly parametrized by

$$\operatorname{Iso}(W, \mathfrak{so}(V)_+) \times \{T : \|T\|_{\operatorname{op}} < 1\}.$$

The negative sector has the identical description with $+$ and $-$ interchanged.

3. In the positive sector, write $B = A_+$ and $T = A_- A_+^{-1}$. The homotopy

$$B_t = B(B^*B)^{-t/2}, \quad T_t = (1-t)T, \quad A_t(\xi) = B_t\xi + T_tB_t\xi$$

is a strong deformation retraction onto the space of orthogonal isomorphisms $W \rightarrow \mathfrak{so}(V)_+$. Every terminal pencil satisfies

$$A_1(\xi)A_1(\eta) + A_1(\eta)A_1(\xi) = -2\langle \xi, \eta \rangle I_V.$$

Hence every positive maximal pencil deforms through elliptic linear pencils to a self-dual Clifford pencil. Every negative maximal pencil similarly deforms to an anti-self-dual Clifford pencil.

4. Each Pfaffian sector has the homotopy type of $O(3)$ and therefore has two path components, distinguished by the orientation of the dominant Hodge isomorphism. The full space of maximal pencils has exactly four path components, each homotopy equivalent to $SO(3)$. After quotienting by the global sign $A \sim -A$, the two orientation components inside each Pfaffian sector are identified. Exactly two path components remain, distinguished by $\text{sgn } q_A$.

Proof. Choose oriented orthonormal quaternionic triples E_1^+, E_2^+, E_3^+ and E_1^-, E_2^-, E_3^- in the two Hodge summands. They satisfy

$$E_i^\pm E_j^\pm + E_j^\pm E_i^\pm = -2\delta_{ij}I, \quad [E_i^+, E_j^-] = 0,$$

and are orthonormal for $\langle \cdot, \cdot \rangle_0$. Write

$$A_+(\xi) = \sum_i a_i E_i^+, \quad A_-(\xi) = \sum_i b_i E_i^-.$$

The standard four-dimensional Pfaffian identity gives

$$\text{Pf}(A(\xi)) = |a|^2 - |b|^2 = r_+(\xi)^2 - r_-(\xi)^2.$$

The two Hodge components commute and square to $-r_+^2 I$ and $-r_-^2 I$. Their product is symmetric, has square $r_+^2 r_-^2 I$, and has trace zero. Therefore its eigenvalues are $\pm r_+ r_-$, each twice, and the eigenvalues of $-A(\xi)^2$ are $(r_+ + r_-)^2$ and $(r_+ - r_-)^2$, each twice. Moreover,

$$(A_+ + A_-)(A_+ - A_-) = A_+^2 - A_-^2 = -q_A I.$$

Thus multiplication by $-(A_+ - A_-)/q_A$ is both a left and a right inverse, which gives the displayed inverse formula. The second expression for $s_{\min}$ is then the difference-of-squares identity.

Since $\det A(\xi) = q_A(\xi)^2$, ellipticity is equivalent to q_A having no nonzero zero. A homogeneous quadratic form with no zero on the connected sphere $S(W)$ has one sign and is definite. If $q_A > 0$, then $\|A_-(\xi)\|_0 < \|A_+(\xi)\|_0$ for every $\xi \neq 0$. Hence A_+ is injective and therefore an isomorphism. Substituting $\xi = A_+^{-1}X$ gives $\|TX\|_0 < \|X\|_0$ for every $X \neq 0$, so $\|T\|_{\text{op}} < 1$. The converse and the negative case are immediate.

For the displayed homotopy, B_t remains an isomorphism and $\|T_t\|_{\text{op}} < 1$, so the graph description keeps q_{A_t} positive. At $t = 1$, B_1 is the orthogonal factor in the polar decomposition of B . An orthogonal isomorphism into either Hodge summand transports the quaternionic relations above and is therefore a Clifford pencil. The homotopy fixes such pencils pointwise.

The strict-contraction ball is convex, while the isomorphism factor retracts onto $O(3)$. Thus each Pfaffian sector has two components. Multiplication by -1 reverses the orientation of a three-dimensional dominant isomorphism but leaves the Pfaffian quadratic form unchanged, proving the final quotient statement. $\square$

Theorem 10 (Pfaffian metric, sharp distortion, and Cartan normal form). *Retain the hypotheses of the maximal four-dimensional pencil theorem and assume $q_A > 0$. Put*

$$B = A_+ : W \longrightarrow \mathfrak{so}(V)_+, \quad T = A_- A_+^{-1}, \quad \tau = \|T\|_{\text{op}} < 1.$$

Define the Pfaffian metric on W by

$$|\xi|_A^2 = q_A(\xi) = \|B\xi\|_0^2 - \|TB\xi\|_0^2.$$

For $\xi \neq 0$, set

$$t_A(\xi) = \frac{\|TB\xi\|_0}{\|B\xi\|_0} \in [0, \tau].$$

Then the following hold.

1. The two repeated singular values have the metric-normalized form

$$s_{\max}(A(\xi)) = |\xi|_A \sqrt{\frac{1+t_A(\xi)}{1-t_A(\xi)}}, \quad s_{\min}(A(\xi)) = |\xi|_A \sqrt{\frac{1-t_A(\xi)}{1+t_A(\xi)}}.$$

In particular,

$$s_{\max}(A(\xi))s_{\min}(A(\xi)) = |\xi|_A^2, \quad \det A(\xi) = |\xi|_A^4.$$

Thus the coefficient normalized by the Pfaffian metric has reciprocal singular values.

2. Let

$$h_A = \operatorname{artanh} \tau.$$

The sharp uniform symbol bounds are

$$e^{-h_A} |\xi|_A \|v\| \leq \|A(\xi)v\| \leq e^{h_A} |\xi|_A \|v\|$$

for all $\xi \in W$ and $v \in V$. Equivalently,

$$\sup_{\xi \neq 0} \operatorname{cond} A(\xi) = e^{2h_A} = \frac{1+\tau}{1-\tau}.$$

Both constants are attained.

3. Set

$$C = (I - T^*T)^{1/2} B : (W, |\cdot|_A) \longrightarrow \mathfrak{so}(V)_+.$$

Then C is an isometry. If $T = U|T|$ is its polar decomposition and $H = \operatorname{artanh} |T|$, then

$$A_+ = \cosh(H)C, \quad A_- = U \sinh(H)C.$$

This is the canonical hyperbolic factorization of the pencil relative to its own Pfaffian metric, and $\|H\|_{\operatorname{op}} = h_A$.

4. Declare two positive pencils equivalent when

$$A'(\xi) = gA(S\xi)g^{-1}$$

for some $S \in GL(W)$ and $g \in SO(V)$. Choose oriented orthonormal quaternionic triples $E_i^\pm$ in the two Hodge summands. Every positive pencil is equivalent to exactly one proper-singular-value normal form

$$A_{\lambda}(\xi) = \sum_{i=1}^3 \xi_i E_i^+ + \sum_{i=1}^3 \lambda_i \xi_i E_i^-$$

with

$$1 > \lambda_1 \geq \lambda_2 \geq |\lambda_3| \geq 0, \quad \lambda_1, \lambda_2 \geq 0.$$

Here $|\lambda_1|, |\lambda_2|, |\lambda_3|$ are the singular values of T , and for full-rank T the sign of λ_3 is $\operatorname{sgn} \det T$. Two positive pencils are equivalent if and only if these signed singular triples agree.

After the covector rescaling

$$y_i = \sqrt{1 - \lambda_i^2} \xi_i, \quad h_i = \operatorname{artanh} \lambda_i,$$

the Pfaffian metric is Euclidean and the normal form becomes

$$A_{\mathbf{h}}(y) = \sum_{i=1}^3 y_i (\cosh h_i E_i^+ + \sinh h_i E_i^-),$$

where

$$h_1 \geq h_2 \geq |h_3|, \quad h_1, h_2 \geq 0.$$

The coefficient on the i th Cartan axis has repeated singular values $e^{|h_i|}$ and $e^{-|h_i|}$. The negative Pfaffian sector has the same statements with the two Hodge summands interchanged.

Proof. Write $r_+ = \|B\xi\|_0$ and $r_- = \|TB\xi\|_0 = t_A(\xi)r_+$. Since $|\xi|_A^2 = r_+^2 - r_-^2 = r_+^2(1 - t_A(\xi)^2)$, substitution into the singular values $r_+ \pm r_-$ gives the first formulas. Their product is the Pfaffian, and the determinant is its square.

The scalar function

$$t \mapsto \sqrt{\frac{1+t}{1-t}}$$

is increasing on $[0, 1)$. Since the operator norm of T is attained in finite dimensions, maximizing $t_A(\xi)$ gives τ . The identities

$$\sqrt{\frac{1+\tau}{1-\tau}} = e^{\operatorname{artanh} \tau}, \quad \frac{1+\tau}{1-\tau} = e^{2\operatorname{artanh} \tau}$$

prove the sharp bounds and condition-number formula.

For the factorization,

$$\|C\xi\|_0^2 = \langle B\xi, (I - T^*T)B\xi \rangle_0 = q_A(\xi),$$

so C is an isometry for the Pfaffian metric. Functional calculus and the polar decomposition give

$$B = (I - |T|^2)^{-1/2}C = \cosh(H)C$$

and

$$TB = U|T|(I - |T|^2)^{-1/2}C = U \sinh(H)C.$$

Moreover $\|H\|_{\text{op}} = \operatorname{artanh} \|T\|_{\text{op}}$.

A base change replaces B by BS and leaves $T = A_- A_+^{-1}$ unchanged. An oriented fiber isometry acts on the Hodge summands through an arbitrary pair in $SO(3)_+ \times SO(3)_-$ and therefore sends T to $R_- T R_+^{-1}$. Proper singular-value decomposition gives

$$R_- T R_+^{-1} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$$

with the displayed chamber conditions; the signed triple is unique. A base change then sets the transformed dominant map equal to the identity. This proves existence and completeness of the first normal form.

Finally,

$$\frac{1}{\sqrt{1-\lambda_i^2}} = \cosh h_i, \quad \frac{\lambda_i}{\sqrt{1-\lambda_i^2}} = \sinh h_i.$$

The rescaling therefore converts the Pfaffian form $\sum_i (1 - \lambda_i^2) \xi_i^2$ to $\sum_i y_i^2$ and yields the Cartan form. On a coordinate axis the two singular values are $\cosh |h_i| \pm \sinh |h_i| = e^{\pm |h_i|}$. $\square$

Remark 3. *The graph-of-a-contraction description and the existence of Cartan coordinates belong to the classical geometry of positive subspaces and noncompact symmetric spaces [6]. The contribution asserted here is the exact translation into maximal skew-adjoint elliptic pencils: the canonical Pfaffian metric, reciprocal repeated singular spectrum, closed inverse, sharp coercivity, signed proper-SVD invariants, hyperbolic factorization, and the condition-number formulas. No broader priority claim about the classical geometry is intended.*

Theorem 11 (Pfaffian-preserving geodesic flattening). *Retain the positive-Pfaffian hypotheses and notation of the preceding theorem. Thus*

$$A_+ = \cosh(H)C, \quad A_- = U \sinh(H)C,$$

where $C : (W, |\cdot|_A) \rightarrow \mathfrak{so}(V)_+$ is an isometry, $H = \operatorname{artanh} |T| \geq 0$, and U is the partial isometry in the polar decomposition of T . For $0 \leq s \leq 1$, define

$$A_{s,+} = \cosh((1-s)H)C, \quad A_{s,-} = U \sinh((1-s)H)C.$$

Let $\varepsilon \in \{+, -\}$ be the orientation sign of the isometry $C : W \rightarrow \mathfrak{so}(V)_+$ relative to the chosen orientations. Orient $\Pi_s = A_s(W)$ through the isomorphism $A_s : W \rightarrow \Pi_s$. Let $\mathcal{G}_+^\varepsilon$ be the connected component of the oriented Pfaffian-positive Grassmannian containing these planes; at $s = 1$, the Hodge plane $\mathfrak{so}(V)_+$ carries the orientation transported by C . Each such component is naturally the symmetric space $SO_0(3, 3)/(SO(3) \times SO(3))$. Equip it with its invariant Riemannian metric, normalized so that a diagonal Cartan flat

$$\mathbf{h} \mapsto \operatorname{span}\{\cosh h_i E_i^+ + \sinh h_i E_i^- : 1 \leq i \leq 3\}$$

has Euclidean metric $dh_1^2 + dh_2^2 + dh_3^2$. If h_1, h_2, h_3 are the signed Cartan rapidities of A , then:

1. The deformation is pointwise Pfaffian-preserving:

$$q_{A_s}(\xi) = q_A(\xi), \quad \det A_s(\xi) = \det A(\xi)$$

for every s and ξ . It remains elliptic, begins at $A_0 = A$, and ends at the positive-chiral Clifford pencil $A_1 = C$.

2. The oriented image planes Π_s form the unique Riemannian geodesic in $\mathcal{G}_+^\varepsilon$ from Π_0 to the dominant Hodge plane $\Pi_1 = \mathfrak{so}(V)_+$ with the orientation transported by C . Its constant speed and length are

$$d_2(\Pi_0, \Pi_1) = (h_1^2 + h_2^2 + h_3^2)^{1/2} = \|H\|_{\text{HS}}.$$

More generally,

$$d_2(\Pi_s, \Pi_1) = (1-s)d_2(\Pi_0, \Pi_1).$$

3. Put

$$d_\infty(A) = \max_i |h_i| = \|H\|_{\text{op}}.$$

Then

$$d_\infty(A) = \operatorname{artanh} \|T\|_{\text{op}} = \frac{1}{2} \log \left(\sup_{\xi \neq 0} \operatorname{cond} A(\xi) \right).$$

The Riemannian and spectral Cartan radii satisfy the sharp comparison

$$d_\infty(A) \leq d_2(\Pi_0, \Pi_1) \leq \sqrt{3} d_\infty(A).$$

The left equality occurs when only one rapidity is nonzero, and the right when all three absolute rapidities are equal.

4. The global distortion improves exactly along the geodesic:

$$\sup_{\xi \neq 0} \text{cond } A_s(\xi) = \exp(2(1-s)d_\infty(A)) = \left(\sup_{\xi \neq 0} \text{cond } A(\xi) \right)^{1-s}.$$

Equivalently, the sharp Pfaffian-metric coercivity constants are

$$\inf_{|\xi|_A=1} s_{\min}(A_s(\xi)) = e^{-(1-s)d_\infty(A)}, \quad \sup_{|\xi|_A=1} s_{\max}(A_s(\xi)) = e^{(1-s)d_\infty(A)}.$$

Thus the logarithmic condition number is affine along the canonical flattening, while the determinant is fixed pointwise.

The negative-Pfaffian sector has the identical statement with the two Hodge summands interchanged.

Proof. Set $K_s = (1-s)H$. Since $\cosh(K_s)$ and $\sinh(K_s)$ are commuting functions of the same self-adjoint operator, and since U^*U is the support projection of H while $\sinh(K_s)$ vanishes on $\ker H$,

$$A_{s,+}^* A_{s,+} - A_{s,-}^* A_{s,-} = C^* (\cosh^2(K_s) - \sinh^2(K_s)) C = C^* C.$$

The right-hand side is the Pfaffian metric of A , proving the first claim and hence pointwise determinant preservation. At $s = 1$ the subordinate Hodge component vanishes and C is an isometry, so A_1 is Clifford.

Choose proper-singular-value coordinates for which $H = \text{diag}(|h_1|, |h_2|, |h_3|)$ and absorb the signs of the h_i into U . Then

$$\Pi_s = \text{span}\{\cosh((1-s)h_i)E_i^+ + \sinh((1-s)h_i)E_i^-\}_{i=1}^3.$$

These planes lie in a maximal flat of the relevant connected component $\mathcal{G}_+^\varepsilon \cong SO_0(3, 3)/(SO(3) \times SO(3))$. Under the stated normalization, straight lines in the rapidity coordinates are geodesics with Euclidean speed. The symmetric space is complete, simply connected, and of nonpositive curvature, so the geodesic between the two planes is unique [6]. This gives the distance and constant-speed formulas.

The subordinate contraction along the path is

$$T_s = U \tanh((1-s)H),$$

so functional calculus gives

$$\|T_s\|_{\text{op}} = \tanh((1-s)\|H\|_{\text{op}}).$$

The sharp condition-number theorem above therefore yields

$$\sup \text{cond } A_s = \frac{1 + \|T_s\|}{1 - \|T_s\|} = e^{2(1-s)\|H\|_{\text{op}}}.$$

The sharp extremal singular-value formulas follow in the same way. Finally, $\|H\|_{\text{op}} \leq \|H\|_{\text{HS}} \leq \sqrt{3}\|H\|_{\text{op}}$, with the stated equality cases. $\square$

Remark 4. *The geodesic description belongs to the classical symmetric-space geometry of positive planes. The operator statement isolated here is the simultaneous preservation of the full Pfaffian quadratic form, exact geodesic optimality, and affine decay of the logarithmic symbol condition number. In particular, the Clifford endpoint is reached without changing the determinant of any nonzero coefficient matrix.*

Corollary 9 (universal maximal architecture on $\mathfrak{u}(2)$). *Every skew-adjoint elliptic linear pencil*

$$A : \mathbb{R}^3 \longrightarrow \mathfrak{so}(\mathfrak{u}(2))$$

—without any equivariance or bilinearity assumption—is elliptically homotopic to a real Clifford pencil. Before quotienting by global sign there are four path components, and after quotienting there are exactly two, detected by the Pfaffian sign. The two balanced trace-completed Pauli symbols constructed above realize the two chiral classes.

Remark 5. *The use of maximal definite three-planes in Λ^2 is classical in four-dimensional geometry; such a plane determines a conformal structure for which it is a self-dual or anti-self-dual summand [9]. The theorem above keeps the original fiber metric fixed and extracts the stronger operator-theoretic data needed here: the exact two singular values, the strict-contraction graph coordinate, the linear Clifford retraction, and the path components of the full maximal-pencil space.*

Corollary 10 (exact bound on a matrix fiber). *A covector-linear skew-adjoint elliptic pencil on $\mathfrak{u}(N)$ over $\mathbb{R}^n$ exists if and only if*

$$n \leq \rho(N^2) - 1.$$

If $a = \nu_2(N)$, then the maximum is

$$n_{\max}(N) = \begin{cases} 4a, & a \text{ even,} \\ 4a - 1, & a \text{ odd.} \end{cases}$$

In particular, $n_{\max} = 0$ for odd N , while

$$n_{\max}(2) = 3, \quad n_{\max}(4) = 8, \quad n_{\max}(8) = 11, \quad n_{\max}(16) = 16.$$

Proof. The real dimension of $\mathfrak{u}(N)$ is N^2 . Write $N = 2^a b$ with b odd. Then $N^2 = 2^{2a} b^2$. If a is even, the Hurwitz–Radon formula gives $\rho(N^2) = 4a + 1$; if a is odd, it gives $\rho(N^2) = 4a$. $\square$

Corollary 11 (constant-coefficient variational elliptic operators). *Let V be a real m -dimensional metric space. A constant-coefficient first-order operator*

$$L = \sum_{j=1}^n A_j \partial_j, \quad A_j^* = -A_j,$$

can be formally self-adjoint and elliptic exactly in the range $n \leq \rho(m) - 1$. In the existence range one may choose the coefficients to satisfy the real Clifford relations, in which case

$$L^2 = -\Delta I_V.$$

On compactly supported fields, or under boundary conditions for which the formal adjoint is realized, such an L is the Hessian of the real quadratic action

$$\mathcal{I}(u) = \frac{1}{2} \int \langle u, Lu \rangle dx.$$

Remark 6. *The dimension theorem is classical and is not claimed as a new result here. Its role is to close the global-realizability question left open by the pointwise centralizer completion and to separate a universal topological restriction from the sharper two-word algebraic obstruction.*

8 Pointwise centralizer completion and sharpness

The preceding section identifies the unique equivariant rank-two trace wedge and proves that it can repair a commutator pencil only on $\mathfrak{u}(2)$. We now separate that global covector-linear classification from the pointwise algebraic question: how much rank is required to repair the entire centralizer of a fixed coefficient J ?

Proposition 4 (rank-minimality of any commutator repair). *Let*

$$z(J) = \dim_{\mathbb{R}} \ker C_J = \sum_{\lambda} m_{\lambda}^2.$$

If an endomorphism R makes $C_J + R$ invertible, then

$$\text{rank } R \geq z(J).$$

Consequently, a rank-two correction can repair a commutator symbol only when $N = 2$ and J is noncentral. In that case αK_{J_0} attains the minimum possible repair rank.

Proof. Rank subadditivity gives

$$\text{rank}(C_J + R) \leq N^2 - z(J) + \text{rank } R.$$

Invertibility forces the right-hand side to be at least N^2 . □

8.1 Optimal completion of the full centralizer

The rank-two contraction above repairs only the plane spanned by the scalar and traceless coefficient directions. The rank bound shows what a complete repair must do: it must act injectively on the entire unitary centralizer. The following pointwise theorem gives the exact optimum.

Let

$$\mathfrak{z}_J = \ker C_J = \{Y \in \mathfrak{u}(N) : [J, Y] = 0\}, \quad z(J) = \dim_{\mathbb{R}} \mathfrak{z}_J = \sum_{\lambda} m_{\lambda}^2,$$

and let Π_J be the orthogonal projection onto $\mathfrak{z}_J$.

Theorem 12 (rank-minimal skew-adjoint centralizer completion). *The parity identity*

$$z(J) \equiv N \pmod{2}$$

holds. If N is odd, no skew-adjoint endomorphism of $\mathfrak{u}(N)$ is invertible.

Assume N is even. Choose any orthogonal complex structure $\mathcal{J}_J$ on $\mathfrak{z}_J$, so

$$\mathcal{J}_J^* = -\mathcal{J}_J, \quad \mathcal{J}_J^2 = -I_{\mathfrak{z}_J},$$

and extend it by zero on $\mathfrak{z}_J^{\perp}$. For $x\alpha \neq 0$, define

$$\mathcal{A}_{x,\alpha,\mathcal{J}}(J) = xC_J + \alpha\mathcal{J}_J\Pi_J.$$

Then $\mathcal{A}_{x,\alpha,\mathcal{J}}(J)$ is skew-adjoint and invertible,

$$\text{rank}(\alpha\mathcal{J}_J\Pi_J) = z(J),$$

and this rank is the smallest possible rank of any additive repair of C_J that can make it invertible.

Proof. Since $m^2 \equiv m \pmod{2}$,

$$z(J) = \sum_{\lambda} m_{\lambda}^2 \equiv \sum_{\lambda} m_{\lambda} = N \pmod{2}.$$

A real skew-adjoint endomorphism of odd-dimensional space has zero determinant, proving the odd- N assertion.

The orthogonal decomposition

$$\mathfrak{u}(N) = \mathfrak{z}_J \oplus \mathfrak{z}_J^{\perp}$$

reduces C_J . It is zero on $\mathfrak{z}_J$ and invertible on $\mathfrak{z}_J^{\perp}$. The completion is $\alpha\mathcal{J}_J$ on the first summand and xC_J on the second, hence is skew-adjoint and invertible. Its rank is $z(J)$. The preceding rank bound shows that no repair of smaller rank can succeed. $\square$

Corollary 12 (exact spectrum, determinant, and inverse). *Let λ range over the distinct eigenvalues of $-iJ$. The singular spectrum of $\mathcal{A}_{x,\alpha,\mathcal{J}}(J)$ consists of*

$$|\alpha| \quad \text{with multiplicity } z(J),$$

and

$$|x| |\lambda - \mu| \quad \text{with multiplicity } 2m_{\lambda}m_{\mu} \quad (\lambda < \mu).$$

Consequently,

$$\left| \det_{\mathbb{R}} \mathcal{A}_{x,\alpha,\mathcal{J}}(J) \right| = |\alpha|^{z(J)} |x|^{N^2 - z(J)} \prod_{\lambda < \mu} |\lambda - \mu|^{2m_{\lambda}m_{\mu}}.$$

If J is noncentral and

$$\delta(J) = \min_{\lambda \neq \mu} |\lambda - \mu|,$$

then

$$s_{\min}(\mathcal{A}_{x,\alpha,\mathcal{J}}(J)) = \min\{|\alpha|, |x|\delta(J)\}.$$

Moreover,

$$\mathcal{A}_{x,\alpha,\mathcal{J}}(J)^{-1} = -\alpha^{-1} \mathcal{J}_J \Pi_J + x^{-1} \left(C_J|_{\mathfrak{z}_J^{\perp}} \right)^{-1} (I - \Pi_J).$$

For central J , the second summand is absent.

Corollary 13 (trace-pair realization and architectural cost). *Let $h_1, \dots, h_{z(J)}$ be an orthonormal basis of $\mathfrak{z}_J$ and pair the basis vectors. Then the optimal completion can be written as the sum of $z(J)/2$ rank-two trace contractions,*

$$R(Y) = \sum_{r=1}^{z(J)/2} \alpha_r (\langle h_{2r}, Y \rangle h_{2r-1} - \langle h_{2r-1}, Y \rangle h_{2r}), \quad \alpha_r \neq 0.$$

Thus the minimal number of independent trace-coupling planes is

$$\frac{1}{2} \sum_{\lambda} m_{\lambda}^2.$$

For a simple spectrum this is $N/2$; for a Clifford coefficient with sign multiplicities p, q , it is $(p^2 + q^2)/2$.

Remark 7. *This theorem is pointwise in J . The projection Π_J and the choice of $\mathcal{J}_J$ need not admit a canonical continuous dependence on a covector-dependent family $J(\xi)$ across spectral collisions, where the centralizer dimension may jump. Therefore the theorem identifies the exact algebraic repair rank, but does not by itself construct a universal differential principal symbol. The Pauli example below remains the explicit linear first-order realization.*

Corollary 14 (explicit trace-completed elliptic Hessian). *Let $1 \leq n \leq 3$, let $\sigma_1, \sigma_2, \sigma_3$ be the Pauli matrices, and set*

$$J(\xi) = i \sum_{a=1}^n \xi_a \sigma_a \in \mathfrak{u}(2).$$

For $x\alpha \neq 0$, the coefficient symbol

$$A(\xi) = x[J(\xi), \cdot] + \alpha K_{J(\xi)}$$

is skew-adjoint and invertible for every $\xi \neq 0$. Its singular values are

$$2|x||\xi| \quad \text{twice}, \quad \sqrt{2}|\alpha||\xi| \quad \text{twice},$$

so its ellipticity margin is

$$|\xi| \min\{2|x|, \sqrt{2}|\alpha|\}.$$

Let L be the constant-coefficient first-order operator with coefficient symbol $A(\xi)$. Then L is formally self-adjoint, and for compactly supported sections it is the Hessian of the real quadratic action

$$\mathcal{I}(u) = \frac{1}{2} \int \langle u, Lu \rangle dx.$$

If $|\alpha| = \sqrt{2}|x|$, then all four singular values equal $2|x||\xi|$, and skew-adjointness yields

$$A(\xi)^2 = -4x^2|\xi|^2 I.$$

Consequently the normalized coefficient matrices $A(e_a)/(2|x|)$, $1 \leq a \leq n$, satisfy the real Clifford relations. When $n = 3$, the construction saturates the Hurwitz–Radon bound

$$3 = \rho(4) - 1.$$

Remark 8. *This exception does not contradict the two-word no-go theorem. The new term uses traces and couples the scalar direction directly to J_0 . It shows that the two-word hypothesis is sharp: trace completion repairs the obstruction in real fiber dimension four, while the rank bound rules out the same rank-two mechanism for every $N \geq 3$.*

9 Scope and computational checks

The first obstruction concerns the positive real Hilbert–Schmidt pairing, an undoubled real field, and the universal two-word family. The equivariant classification removes the two-word restriction while retaining real bilinearity and simultaneous unitary-conjugation symmetry. Coefficient–subspace rigidity further requires adjoint parity only on the actual linear coefficient image. Within that class, the trace wedge is the unique extra skew-adjoint channel and the only elliptic case is the rank-two architecture classified above. The optimal full-centralizer repair is pointwise and requires a choice of complex structure on the centralizer; no covector-linear or globally smooth choice is asserted.

Non-bilinear laws, non-equivariant intertwiners, nonlinear coefficient maps, complex, indefinite, or symplectic pairings, doubled fields, constrained systems, and higher-order symbols remain outside the classification.

Ancillary scripts provide independent exact symbolic checks of the channel basis, restricted adjoint equations, arbitrary-coefficient spectra, trace-completion formulas, quaternionic normal forms, Hodge decomposition, Cartan factorization, and the Pfaffian-preserving geodesic. They reconstruct the relevant matrices from the defining operations rather than importing the closed-form conclusions from the manuscript. Seeded finite-dimensional checks use fixed random seeds; the release harness fixes the Python hash seed and requests single-threaded BLAS execution. These computations are supplementary verification only: every theorem used here is proved analytically in the text, and the scripts do not replace Adams’s theorem or the cited classical symmetric-space geometry. Specialist literature review remains necessary before any assertion of external priority.

A Standard exits from the obstruction

The obstruction is tied to the undoubled first-order real architecture. The following familiar constructions are recorded only to delimit the theorem; they are not part of the classification result.

Proposition 5 (multiplier doubling). *Let $L : E \rightarrow F$ be a first-order operator between equal-rank real metric bundles, and use ellipticity in the classical invertible-symbol sense. On $E \oplus F$, define*

$$\mathcal{H}_L = \begin{pmatrix} 0 & L^* \\ L & 0 \end{pmatrix}.$$

Then $\mathcal{H}_L$ is formally self-adjoint and elliptic if and only if L is elliptic. Moreover,

$$\mathcal{H}_L^2 = \begin{pmatrix} L^*L & 0 \\ 0 & LL^* \end{pmatrix}, \quad \ker \mathcal{H}_L = \ker L \oplus \ker L^*.$$

Proposition 6 (norm-square action). *Suppose $F(q) = 0$ is a first-order equation with linearization L at a solution. The action*

$$I_2(q) = \frac{1}{2} \|F(q)\|^2$$

has Hessian L^*L at that solution. If L is elliptic, then L^*L is a second-order elliptic operator with principal symbol $\sigma(L)^*\sigma(L)$.

These constructions restore variationality by changing the field content or the differential order. They do not contradict the theorem.

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